A PROBLEM IN CHANCE

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A problem in the elementary theory of probabilities, the result of which will interest most students of "College Algebra," is the following:

Two players of unequal skill, A and B, play tennis. To which of them is it of advantage that deuce sets be played?

The discussion is based on the two fundamental principles for finding the probabilities of compound events: *

I. If the chances of the occurrence of two independent events are p_1 and p_2 respectively, the chance that both will occur is the product p_1p_2.

II. If the chances of the occurrence of two mutually exclusive events are p_1 and p_2 respectively, the chance that one of the events will occur is the sum $p_1 + p_2$.

We shall suppose that A's chance of winning each game is fixed and equal to p. Then B's chance of winning each game is $1 - p$, and we write $1 - p = q$. This assumption is artificial but may fairly be made, and p may be chosen as the ratio of the number of games won by A to the whole number played, when the latter number is large.

We may solve the question proposed by finding A's chances of winning the set by each of the two methods of scoring, and then comparing the results.

The chance that A win the set "six-love" is, by I, equal to p^6.

The chance that he win "six-one" we find as follows: of the first six games B wins one, and A five, then A wins the seventh; the first six games may be arranged in six ways, for B may win any one of them. The chance of the occurrence of any particular one of these arrangements is p^6q. Since the different arrangements are mutually exclusive, the chance of the occurrence of one, hence that A win "six-one," is $6p^6q$.

In the same way we find that A's chances of winning with a score of "six-two," "six-three," "six-four" and "six-five" are respectively $21p^6q^2$, $56p^6q^3$, $126p^6q^4$ and $252p^6q^5$. These results are easily found if we remember that A must win the last game, that his other five games and the games won by

* See *e. g.*, Wentworth, *College Algebra*, revised edition, pp. 282, 283.

B may be arranged in any order, and that the different arrangements are mutually exclusive events.

Let us write

$$M = p^6 (1 + 6q + 21q^2 + 56q^3 + 126q^4)$$
$$Np = 252p^6q^5.$$

Then if deuce sets are not played, since winning by different scores gives again mutually exclusive events, A's chance of winning the set is

$$M + Np. \tag{1}$$

If deuce sets are to be played the result is quite different. In this case A may still win the set with B's score less than five games, and for this the chance is, as before, equal to M. But if B wins five games the score must become "five-all," for which event the chance is N. If A wins the next two games, for which the chance is p^2, he wins the set at "seven-five." The chance that he win by this score is then Np^2. If A wins the set, but not the next two games after "five-all," the score must become "six-all." The chance that A and B win each one of the two games after "five-all" is $2pq$. Then the chance that A win the set at "eight-six" is $Np^2\, 2pq$. Similarly we may find the chance that A win at "nine-seven" is $Np^2\,(2pq)^2$. Hence, A's chance to win the set is

$$M + Np^2\,[1 + 2pq + (2pq)^2 + (2pq)^3 \cdots].$$

The geometric series is convergent; for $2pq = 2p(1-p) < \frac{1}{2}$, since $p \neq \frac{1}{2}$. Hence, finally, we may write A's chance of winning, when deuce sets are to be played, as

$$M + \frac{Np^2}{1 - 2pq}. \tag{2}$$

This chance is greater than his chance to win when deuce sets are not played, if

$$\frac{p^2}{1 - 2pq} > p,$$

that is, if $p > 1 - 2p(1-p)$, since $q = 1 - p$,

or
$$(2p-1)(1-p) > 0,$$

or
$$1 > p > \tfrac{1}{2}.$$

Thus the advantage of playing deuce sets lies with A if he is the better player.

As a numerical case, let A's chance of winning a single game be $p = 0.63$. Then A's chance of winning a particular set in a tournament in which deuce sets are not to be played is 0.815, while his chance of winning a particular set in a tournament in which deuce sets are to be played is 0.835. This example is chosen so that the difference between the chances in the two kinds of tournaments shall be a maximum.*

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* Thus, the difference in question is

$$D = 252p^6q^5\left[\frac{p}{1-2pq} - 1\right] = \frac{63}{512}\,\frac{x(1-x^2)^6}{1+x^2},$$

where $x = p - q$, and the value of x which makes D a maximum is found to be a root of the equation $11x^4 + 14x^2 - 1 = 0$, namely $x = 0.26 +$. Then

$$p = \frac{1}{2} + \frac{x}{2} = 0.63 + \quad \text{and} \quad q = \frac{1}{2} + \frac{x}{2} = 0.37 -,$$

and the corresponding maximum value of D is $0.02 -$.